On Homogeneous Distributed Parameter Systems

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***Abstract*—A geometric homogeneity is introduced for evolution equations in a Banach space. Scalability property of solutions of homogeneous evolution equations is proven. Some qualitative characteristics of stability of trivial solution are also provided. In particular, finite-time stability of homogeneous evolution equations is studied. Theoretical results are illustrated on important classes of partial differential equations.**

I. Introduction

The homogeneity is a sort of symmetry, when an object remains consistent (in some sense) with respect to a scaling operation (dilation). In the context of ordinary differential equations and inclusions one encounters three types of homogeneity:

- *the standard homogeneity* (L. Euler in the 17th century, V. Zubov [1], W. Hahn [2]) operates with uniform dilations such as $x \to \lambda x$, where $\lambda > 0$ is a real number and x is an element of a real linear space;
- *the weighted homogeneity* (V. Zubov [3], H. Hermes [4], L. Rosier [5], G. Folland [6]) uses non-uniform (anisotropic) scalings like
$$(x_1, x_2, ..., x_n) \to (\lambda^{r_1} x_1, \lambda^{r_2} x_2, ..., \lambda^{r_n} x_n),$$
where λ and r_i are positive reals, x_i is an element of a real linear space, $i = 1, 2, ..., n$;
- *the geometric homogeneity* (V. Khomenyuk [7], L. Rosier [8], M. Kawski [9]) considers some generalized dilations of one vector field with respect to another one.

Homogeneity is a useful tool for advanced analysis of nonlinear dynamic systems. For instance, it allows local properties of a system (e.g. asymptotic stability) to be extended globally. Qualitative stability analysis of homogeneous systems can be enhanced by means of investigation of homogeneity degree of an asymptotically stable system, for example, the negative degree corresponds to finite-time stability [1], [10], [11], [12], [13], [14]. The control theory applies homogeneous feedbacks for fast robust stabilization (see, [15], [11], [13], [14], [16]) and homogeneous dynamic observers for non-asymptotic state estimation [17], [18]. Homogeneity provides simple algorithms for robustness analysis of nonlinear control systems (in the context of Input-to-State Stability see, for example, [19], [20], [21]). Local homogeneity and homogeneous approximations [3], [4], [14], [22] are considered as a way for simplification of qualitative analysis of the essentially nonlinear dynamic systems. It is worth stressing that investigations of quantitative characteristics (for example, estimation of the settling time for finite-time stable system) require conventional tools (e.g. Lyapunov function method). However, it is well known [3], [5] that any stable homogeneous system always has a homogeneous Lyapunov function. This property simplifies the construction of concrete explicit [23], [24] and implicit Lyapunov functions [25] for homogeneous systems.

The analysis of evolution equations with homogeneous (with respect to uniform scaling) operator was given in [26]. Some important regularizing effects of homogeneity have been discovered in the mentioned paper. Weighted homogeneous evolution equations have been studied in [27], [28]. The most of results for such systems are devoted to the integrability analysis. The elements of the theory of sub-elliptic operators on stratified nilpotent Lie groups is developed in [6] based on a version of the weighted homogeneity. To the best of our knowledge, stability properties (in particular, finite-time stability) of evolution equation have not been studied using homogeneity framework. In the same time, these issues are very important for control and estimation problems of distributed parameters systems [29].

This technical note studies a certain analog of geometric homogeneity, which has not been studied before for infinite dimensional systems. It introduces **d**-homogeneous operators, where **d** denotes a group of homogeneous dilations in a Banach space, which must be agreed with the domain of the operator. The paper demonstrates that the main features of the homogeneous systems, which are important for control design in finite-dimensional setting (like stability and scalability properties of solutions), hold for **d**-homogeneous evolution equations in a Banach space. The paper also shows that **d**-homogeneity can be established for many well-known partial differential equations like KdV, Saint-Venant and Fast Diffusion equations considered in the paper. The presented results can be utilized in the future for extension of homogeneity-based feedback control design tools to infinite dimensional system.

The paper is organized as follows. Model description and basic assumptions are given in the section 2. The section 3 introduces the notion of homogeneous evolution equation in Banach space. It presents the generalized dilation group, the homogeneous set and the homogeneous operator, and studies their properties. Finally, some concluding remarks are given at the end of the paper.

Notation. The paper uses the following standard notation: $\mathbb{R}$ is the field of real numbers and $\mathbb{R}_+ = [0, +\infty)$; $\mathbf{B}$ is a real Banach space with a norm $\|\cdot\|$; $S = \{u \in \mathbf{B} : \|u\| = 1\}$ is the unit sphere in $\mathbf{B}$; $\bar{\Omega}$ denotes the closure of the set $\Omega \subset \mathbf{B}$; $\partial\Omega$ denotes the boundary of the set $\Omega \subset \mathbf{B}$.

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II. Model Description

Let us consider the nonlinear evolution equation

$$\dot{u}(t) = f(u(t)), \quad t \in \mathbb{R}_+, \quad u(t) \in \Omega \subset \mathbf{B} \tag{1}$$

with the initial condition

$$u(0) = \varphi \in \Omega, \tag{2}$$

where the dot denotes a time derivative, the operator $f : D \subset \mathbf{B} \to \mathbf{B}$ has the domain D and $\Omega \subset D$.

Recall that evolution equation may describe finite or infinite dimensional dynamic system. The set Ω restricts admissible solutions. For example, if it is a positive cone (i.e. $\Omega = \mathbb{R}^n_+$), the evolution equation must describe a positive system. In the case of a partial differential equation the set Ω can be utilized in order to possess boundary conditions.

We assume that the Cauchy problem (1), (2) has a solution for any $\varphi \in \Omega$, i.e. there exists a continuous function $u : [0, T) \to \Omega$ defined at least locally ($0 < T \leq +\infty$), which satisfies (in some sense) the equation (1) and the initial condition (2). We do not specify the type of solution. It can be classical (C^1), strong, weak or mild solution. Proofs of main results are given for mild solutions. The extension to other types of solutions is straightforward. The same results can be provided for evolution inclusions.

In general, we do not assume that the Cauchy problem is well-posed, i.e. the solutions may be non-unique and/or they may depend discontinuously on the initial condition. Certainly, the results to be obtained also hold for well-posed models. Existence of solutions and well-posedness of the Cauchy problem like (1), (2) are discussed in literature for different types of operators f and Banach spaces $\mathbf{B}$, see e.g. [30], [31], [32]. These problems go out of the scope of this paper. The main aim is to extend the concept of homogeneous systems to the general evolution equations and to transfer some useful properties and tools, which are derived for homogeneous ordinary differential equations, to a more general disturbed dynamical models.

III. Homogeneous Evolution Equations

The evolution equation (1) is uniquely identified by the operator f and the set Ω. Having the same operator $f : D \subset \mathbf{B} \to \mathbf{B}$ for different sets $\Omega_1 \subset D$ and $\Omega_2 \subset D$, $\Omega_1 \neq \Omega_2$ we may obtain different behaviors of solutions of evolution equations. Therefore, in the context of the evolution equation the homogeneity properties must be studied for both the operator f and the set Ω.

A. Homogeneous Sets in Banach Spaces

Let $\mathcal{L}(\mathbf{B})$ be the space of linear bounded operators $\mathbf{B} \to \mathbf{B}$ equipped with the norm: $\|g\|_{\mathcal{L}} = \sup_{u \in S} \|g(u)\|$ for $g \in \mathcal{L}(\mathbf{B})$.

Definition 1: A map $\mathbf{d} : \mathbb{R} \to \mathcal{L}(\mathbf{B})$ is called **dilation** in the space $\mathbf{B}$ if it satisfies

- **the semigroup property:** $\mathbf{d}(0) = I \in \mathcal{L}(\mathbf{B})$ and $\mathbf{d}(t + s) = \mathbf{d}(t)\mathbf{d}(s)$ for $t, s \in \mathbb{R}$;
- **the strong continuity property:** the map $\mathbf{d}(\cdot)u : \mathbb{R} \to \mathbf{B}$ is continuous for any $u \in \mathbf{B}$;
- **the limit property:** $\lim\limits_{s \to -\infty} \|\mathbf{d}(s)u\| = 0$ and $\lim\limits_{s \to +\infty} \|\mathbf{d}(s)u\| = \infty$ uniformly on $u \in S$.

The dilation $\mathbf{d}$ is a strongly continuous group, since the limit property implies uniqueness of the identity element (see, Proposition 1 given below). The group $\mathbf{d}$ has similar topological characterization as a dilation mappings in Banach (or Frechet) spaces. Indeed, the limit property given above can be interpreted as a version of the Teresaka's condition (see, for example, [33]).

Proposition 1: If $\mathbf{d}$ is a dilation then

1) $\|\mathbf{d}(s)\|_{\mathcal{L}} \neq 0$ for $s \in \mathbb{R}$;
2) $\inf\limits_{u \in S} \|\mathbf{d}(s)u\| < 1$ for $s < 0$ and $\|\mathbf{d}(s)\|_{\mathcal{L}} = \sup\limits_{u \in S} \|\mathbf{d}(s)u\| > 1$ for $s > 0$;
3) $\mathbf{d}(s) \neq I \in \mathcal{L}(\mathbf{B})$ for $s \neq 0$.

Proof: 1) The positivity of the operator norm of the dilation immediately follows from the semigroup property. Indeed, if there exists $s_0 \in \mathbb{R}$ such that $\|\mathbf{d}(s_0)\|_{\mathcal{L}} = 0$ then $\mathbf{d}(s_0) = 0 \in \mathcal{L}(\mathbf{B})$ and $I = d(0) = d(s_0 - s_0) \neq d(-s_0)d(s_0) = 0$.

2) Let us consider, initially, the case $s \in \mathbb{R}_+$. Suppose the contrary, i.e. there exists $s_0 \in \mathbb{R}_+$ such that $\|\mathbf{d}(s_0)\|_{\mathcal{L}} \leq 1$. On the one hand, due to the limit property we have $\lim\limits_{n \to +\infty} \|\mathbf{d}(ns_0)u\|_{\mathcal{L}} = +\infty$ for any $u \in S$. On the other hand,

$$\|\mathbf{d}(ns_0)u\| = \|\mathbf{d}(s_0 + (n-1)s_0)u\| = \|\mathbf{d}(s_0)\mathbf{d}((n-1)s_0)u\| \leq$$
$$\|\mathbf{d}(s_0)\|_{\mathcal{L}} \cdot \|\mathbf{d}((n-1)s_0)u\| \leq \ldots \leq \|u\| = 1,$$

i.e. the sequence $\|\mathbf{d}(ns_0)u\|$ is bounded for any n and we obtain the contradiction.

Consider the case $s < 0$ and suppose the contrary, i.e. there exists $s_0 \in \mathbb{R}_+$ such that $\|\mathbf{d}(-s_0)u\| \geq 1$ for any $u \in S$. If $u_0 \in S$ then

$$u_n = \frac{\mathbf{d}(-s_0)u_{n-1}}{\|\mathbf{d}(-s_0)u_{n-1}\|} \in S, \quad n \geq 1.$$

So, we have

$$1 \leq \|\mathbf{d}(-s_0)u_n\| = \left\| \frac{\mathbf{d}(-2s_0)u_{n-1}}{\|\mathbf{d}(-s_0)u_{n-1}\|} \right\| \leq \|\mathbf{d}(-2s_0)u_{n-1}\| \leq$$
$$\ldots \leq \|\mathbf{d}(-(n+1)s_0)u_0\|.$$

Therefore, $\|\mathbf{d}(-(n+1)s_0)u_0\| \geq 1$ for any $n \geq 1$ and we derive the contradiction to the limit property.

3) Suppose the contrary, i.e. there exists $s' \in \mathbb{R}\backslash\{0\}$ such that $\mathbf{d}(s') = I$. Then the semigroup property implies $\mathbf{d}(ns') = I$ and $\|\mathbf{d}(ns')u\| = 1$ for any $n = 1, 2, \ldots$ and any $u \in S$. This contradicts the limit property. ■

The next definition introduces domains of evolution operators to be studied in this paper. The domain should be invariant with respect to dilation.

Definition 2: A nonempty set Ω is said to be $\mathbf{d}$-homogeneous iff $\mathbf{d}(s)\Omega \subseteq \Omega$ for any $s \in \mathbb{R}$, where $\mathbf{d} : \mathbb{R} \to \mathcal{L}(\mathbf{B})$ is a dilation in $\mathbf{B}$.

A $\mathbf{d}$-homogeneous set Ω becomes a homogeneous space [34], if the dilation $\mathbf{d}$ is a transitive group on Ω. The operators $\mathbf{d}(s)$ for $s \in \mathbb{R}$ are called symmetries in this case.

Let us give some examples of **d**-homogeneous sets and their dilations.

- *Uniform dilation* (L. Euler):
  $\Omega = \mathbb{R}^n$, $\mathbf{d}(s) = e^s$.
- *Weighted dilation* (V. Zubov [3]):
  $\Omega = \mathbb{R}^n$, $\mathbf{d}(s) = \mathrm{diag}\{e^{r_i s}\}$, $r_i > 0$, $i = 1, 2, ..., n$.
- *Geometric dilation* (L. Rosier [8], M. Kawski [9])
  $\Omega = \mathbb{R}^n$, $\mathbf{d}$ is the flow of an Euler vector field[1].
- *Generalized dilation in a Banach space*:
  $\Omega = \{u \in C([0,p],\mathbb{R}) : u(0) = u^2(p)\}$ and $(\mathbf{d}(s)u)(x) = e^{s-0.5sx/p}u(x)$, where $x \in [0,p]$. Indeed, $(\mathbf{d}(s)u)(0) = e^s u(0)$ and $(\mathbf{d}(s)u)(p) = e^{0.5s}u(p)$ imply $(\mathbf{d}(s)u)(0) = [(\mathbf{d}(s)u)(p)]^2$ for any $u \in \Omega$ and any $s \in \mathbb{R}$. *Such dilations have never been studied before in the context of homogeneous systems.*

Note that ***d****-homogeneity can be considered as an analog of geometric homogeneity known for finite-dimensional vector fields. Indeed, in a particular case, a group* ***d*** *may be generated by some evolution operator in the Banach space* ***B***.

The set

$$S_{\mathbf{d}}(r) = \{u \in \Omega : \|\mathbf{d}\,(\ln(r))\,u\| = 1\}\,, \quad r > 0 \tag{3}$$

is called **the homogeneous sphere** of the radius r.

Homogeneous sets and spheres have some useful properties to be utilized below for analysis of evolution equation (1).

Proposition 2: If **d**-homogeneous set Ω is non-trivial[2] then

1) the set $S_{\mathbf{d}}(1)$ is non-empty and for any $u \in \Omega\backslash\{0\}$ there exists $u_0 \in S_{\mathbf{d}}(1)$ such that $u = d(s)u_0$ for some $s \in \mathbb{R}$;
2) $S_{\mathbf{d}}(r) = \mathbf{d}(\ln(1/r))S_{\mathbf{d}}(1)$ for $r > 0$;
3) $\sup\limits_{y\in S_{\mathbf{d}}(r)} \|y\| \to 0$ as $r \to 0$;
4) $\Omega\backslash\{0\} = \bigcup\limits_{r>0} S_{\mathbf{d}}(r)$.

Proof: 1) Let u be an arbitrary element of the nonempty set $\Omega\backslash\{0\}$. If $\|u\| = 1$ then $u \in S_{\mathbf{d}}(1)$. Since the group $\mathbf{d}$ is strongly continuous then the limit property implies that for $\|u\| > 1$ there exists $s < 0 : \|\mathbf{d}(s)u\| = 1$, and for $0 < \|u\| < 1$ there exists $s > 0 : \|\mathbf{d}(s)u\| = 1$. Since $d(s)u \in \Omega$ for any $s \in \mathbb{R}$ then $d(s)u \in S_{\mathbf{d}}(1)$, i.e. $S_{\mathbf{d}}(1) \neq \emptyset$.

2) On the one hand, $y \in S_{\mathbf{d}}(r)$ means that there exists $u \in S_{\mathbf{d}}(1) \subseteq S$ such that $y = \mathbf{d}(\ln(r))u$. On the other hand, $u \in S_{\mathbf{d}}(1)$ means that $1 = \|u\| = \|d(0)u\| = \|\mathbf{d}(\ln(r) - \ln(r))u\| = \|\mathbf{d}(\ln(r))\mathbf{d}(\ln(1/r))u\|$, i.e. $\mathbf{d}(\ln(1/r))u \in S_{\mathbf{d}}(r)$.

3) The limit property guarantees $\|\mathbf{d}(\ln(r))u\| \to 0$ as $r \to 0$ uniformly on S. This immediately implies the third claim.

4) If $u \in \Omega\backslash\{0\}$ and $u_0 \in S_{\mathbf{d}}(1)$ is such that $\mathbf{d}(s)u = u_0$ for some $s \in \mathbb{R}$ then the semigroup property guarantees $\mathbf{d}(-s)\mathbf{d}(s)u = \mathbf{d}(-s)u_0$ or equivalently $u = \mathbf{d}(-s)u_0$. Therefore, $u \in S_{\mathbf{d}}(r)$ with $r = e^{-s}$, i.e. each element from $\Omega\backslash\{0\}$ belongs to a homogeneous sphere. ■

B. Homogeneous Operators and Equations

The definition given below presents the class of operators to be studied in this paper. It utilizes the conventional identity (4) in order to introduce the homogeneity relation (see, for example, [35]).

Definition 3: An operator $f : D \subset \mathbf{B} \to \mathbf{B}$ is said to be **d**-homogeneous of degree ν on the set $\Omega \subset D$ if Ω is **d**-homogeneous and

$$f(\mathbf{d}(s)u) = e^{\nu s}\mathbf{d}(s)f(u) \qquad s \in \mathbb{R}, \quad u \in \Omega, \tag{4}$$

where $\mathbf{d}$ is a dilation in $\mathbf{B}$ and $\nu \in \mathbb{R}$.

The evolution equation (1) is said to be **d**-homogeneous on Ω iff the corresponding operator $f : D \subset \mathbf{B} \to \mathbf{B}$ is **d**-homogeneous on Ω.

Homogeneity can be discovered in many physical models. Examples of homogeneous ordinary differential equations can be found in the literature, see e.g. [3], [12], [17], [13]. Let us consider two examples of homogeneous partial differential equations, which appear in mathematical physics.

• **Korteweg-de Vries equation** (KdV equation) is the homogeneous partial differential equation ([36], [28]):

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial x^3} - u\frac{\partial u}{\partial x},$$

where u is a scalar function of time $t \in \mathbb{R}_+$ and space $x \in \mathbb{R}_+$ variables. KdV equation describes waves on shallow water surfaces. Let the boundary condition has the form $u\big|_{x=0} = 0$. The operator $f : C^3(\mathbb{R}_+,\mathbb{R}) \to C(\mathbb{R}_+,\mathbb{R})$ defined by $f(u) = -u''' - uu'$ for $u \in C^3(\mathbb{R}_+,\mathbb{R})$ is **d**-homogeneous of degree $\nu = 3$ on $\Omega = \left\{z \in C^3(\mathbb{R}_+,\mathbb{R}) : z'(0) = 0\right\}$ with the dilation group defined by $(\mathbf{d}(s)u)(x) = e^{2s}u(e^s x)$, where $x \in \mathbb{R}_+$, $u \in C(\mathbb{R}_+,\mathbb{R})$ and $s \in \mathbb{R}$ is the dilation argument. Obviously, $\mathbf{d}(s)\Omega \subseteq \Omega$. Due to $(d(s)u')(x) = e^{2s}u'(y)\big|_{y=e^s x}$ and $[d(s)u]'(x) = [e^{2s}u(e^s x)]' = e^{3s}u'(y)\big|_{y=e^s x}$ for $x \in \mathbb{R}_+$ we derive

$$\begin{aligned}&[f(d(s)u)](x) = -\left[e^{2s}u(e^s x)\right]''' - e^{2s}u(e^s x)[e^{2s}u(e^s x)]' \\ &= -e^{5s}u'''(y) - e^{5s}u(y)u'(y)\big|_{y=e^s x} = [e^{3s}d(s)f(u)](x),\end{aligned}$$

for any $s \in \mathbb{R}$.

• The **Saint-Venant equation** is an example of a system of conservation laws studied in [37]. In the field of hydraulics, it represents the flow in open-channels by the following model

$$\begin{aligned}\frac{\partial H}{\partial t} &= -\frac{\partial}{\partial x}(HV), \\ \frac{\partial V}{\partial t} &= -\frac{\partial}{\partial x}\left(\tfrac{1}{2}V^2 + gH\right),\end{aligned} \tag{5}$$

where H and V are scalar functions of time and space variables. The quantity $H(t,x)$ is the water level at the instant of time $t \in \mathbb{R}_+$ in the point $x \in \mathbb{R}$, and $V(t,x)$ is the water velocity in the same position. The parameter g denotes the gravitation constant. Let us consider the case when the water channel is supported by two overflow spillways (Figure 1), which adjust an input and output flows in a pool (between spillways). The space argument is restricted on the segment $[0,1]$, where $x=0$ and $x=1$ are positions of spillways, and the equation (5) is supported with the boundary conditions [37]:

$$\begin{aligned}H(t,0)V(t,0) - (Z_0 - L_0)^{3/2} = 0,\\ H(t,1)V(t,1) - (H(t,1) - L_1)^{3/2} = 0,\end{aligned}$$

where Z_0 is the water level above the pool and L_0, L_1 are spillways.

[1]A C^1 vector field $\nu : \mathbb{R}^n \to \mathbb{R}^n$ is called Euler if it is complete and $-\nu$ is globally asymptotically stable.

[2]The set Ω is non-trivial if it contains some elements different from 0.

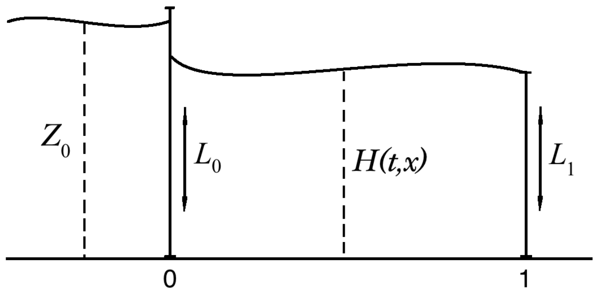

Fig. 1. Water channel with two spillways

Let us show that for $L_0 = Z_0$ and $L_1 = 0$ the corresponding evolution equation is homogeneous. Let us consider the operator $f : D \to C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ defined on $D = C^1([0,1],\mathbb{R}_+) \times C^1([0,1],\mathbb{R})$ by

$$f(u) = \begin{pmatrix} -\frac{\partial}{\partial x}(u_1 u_2) \\ -\frac{\partial}{\partial x}\left(g u_1 + \frac{1}{2}u_2^2\right) \end{pmatrix},$$

where $u = (u_1, u_2) \in D$. The operator f is $\mathbf{d}$-homogeneous of degree $\nu = 1$ on the set

$$\Omega = \left\{ u = (u_1, u_2) \in D : \begin{array}{c} u_1(0)u_2(0) = 0; \\ u_1(1)u_2(1) = u_1^{3/2}(1) \end{array} \right\}$$

with respect to the weighted dilation $\mathbf{d}(s)u = (e^{2s}u_1, e^s u_2)$, where $u = (u_1, u_2) \in C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ and $s \in \mathbb{R}$. Indeed,

$$f(\mathbf{d}(s)u) = \begin{pmatrix} -\frac{\partial}{\partial x}(e^{2s}u_1 e^s u_2) \\ -\frac{\partial}{\partial x}\left(g e^{2s}u_1 + \frac{1}{2}[e^s u_2]^2\right) \end{pmatrix} =$$

$$\begin{pmatrix} -e^{3s}\frac{\partial}{\partial x}(u_1 u_2) \\ -e^{2s}\frac{\partial}{\partial x}\left(g u_1 + \frac{1}{2}u_2^2\right) \end{pmatrix} = e^s d(s) f(u).$$

Finally, the equality $(d(s)u)(x) = (e^{2s}u_1(x), e^s u_2(x)), x \in \mathbb{R}$ implies that for any $u \in \Omega$ one has $d(s)u \in \Omega$, i.e. the set Ω is $\mathbf{d}$-homogeneous.

Remark 1: If the operator $g : D \subset \mathbf{B} \to \mathbf{B}$ is $\mathbf{d}$-homogeneous, then the set $\mathrm{Ker}(g) = \{u \in D : g(u) = 0\}$ is $\mathbf{d}$-homogeneous. Indeed, if $u \in \mathrm{Ker}(g)$ then $g(\mathbf{d}(s)u) = e^{\nu s}\mathbf{d}(s)g(u) = 0$, i.e. $\mathbf{d}(s)u \in \mathrm{Ker}(g)$. So, the kernel of homogeneous operator is $\mathbf{d}$-homogeneous set.

C. Properties of Solutions to Homogeneous Evolution Equations

Homogeneity may simplify a qualitative analysis of partial differential equations. This subsection studies some properties of solutions to homogeneous evolution equations. The next theorem provides the most important scalability property of solutions. Its proof is given for mild solutions of (1), (2), i.e. $u(t,\varphi) = \lim\limits_{\varepsilon \to 0} u^{\varepsilon}(t,\varphi)$ uniformly on t, where $u^{\varepsilon}(t,\varphi)$ is a so-called ε-solution:

$$u^{\varepsilon}(0,\varphi) = u_0 = \varphi, \;\; u^{\varepsilon}(t,\varphi) = u_i \in \Omega \;\text{ for }\; t \in [t_i, t_{i+1})$$

$$\frac{u_{i+1} - u_i}{t_{i+1} - t_i} = f(u_i) \;\text{ for }\; i = 0, 1..., k-1, \; t_k = T,$$

where $t_{i+1} - t_i \le \varepsilon$ for $i = 0, ..., k-1$. The proof can be repeated for classical solutions, strong or weak solutions.

Theorem 1 (On Homogeneous Dilation of Solutions): Let an operator $f : \Omega \subset \mathbf{B} \to \mathbf{B}$ be $\mathbf{d}$-homogeneous of degree $\nu \in \mathbb{R}$ and $u(\cdot,\varphi) : [0,T) \to \Omega$ be a solution of the Cauchy problem (1), (2).

Then for any $s \in \mathbb{R}$ the function $u_s : \left[0, \frac{T}{e^{\nu s}}\right) \to \Omega$ defined by the equality $u_s(\tau) = \mathbf{d}(s)u(e^{\nu s}\tau, \varphi)$ is a solution of the evolution equation (1) with the initial condition $u(0) = \mathbf{d}(s)\varphi$.

Proof: Let $u^{\varepsilon}(\cdot,\varphi)$ be an arbitrary ε-solution of the Cauchy problem (1), (2). For an arbitrary $s \in \mathbb{R}$ let us construct a function $u_s^{\varepsilon} : \left[0, \frac{T}{e^{\nu s}}\right) \to \Omega$ using the following relation $u_s^{\varepsilon}(\tau) = d(s)u^{\varepsilon}(e^{\nu s}\tau, \varphi)$, where $\tau \in \left[0, \frac{T}{e^{\nu s}}\right)$.

Let us denote $\tau_i = e^{-\nu s}t_i$ and $u_{s\,i} = \mathbf{d}(s)u_i \in \Omega$. On the one hand, we have $\frac{u_{s\,i+1} - u_{s\,i}}{\tau_{i+1} - \tau_i} = e^{\nu s}\mathbf{d}(s)\frac{u_{i+1} - u_i}{t_{i+1} - t_i}$. On the other hand, the homogeneity of the operator f provides $f(\mathbf{d}(s)u_i) = f(u_{s\,i}) = e^{\nu s}\mathbf{d}(s)f(u_i)$. Hence, we derive

$$\frac{u_{s\;i+1} - u_{s\;i}}{\tau_{i+1} - \tau_i} = f(u_{s\;i}),$$

i.e. the function u_s^{ε} is ε-solution of the evolution equation (1) with the initial condition $u(0) = \mathbf{d}(s)\varphi$.

Finally, since $\|\mathbf{d}(s)\| < +\infty$ for any $s \in \mathbb{R}$, then the inequality $\|u_s^{\varepsilon}(\tau) - u_s(\tau)\| \le \|\mathbf{d}(s)\|_{\mathcal{L}} \cdot \|(u^{\varepsilon}(t) - u(t))\|$ implies that $u_s(\tau) = \lim_{\varepsilon \to 0} u_s^{\varepsilon}(\tau)$ uniformly on τ. ■

Note that under conditions of Theorem 1 the function $u_s^{T'} : \left[0, \frac{T-T'}{e^{\nu s}}\right) \to \Omega$ defined by $u_s^{T'}(\tau) = d(s)u(T' + e^{\nu s}\tau, \varphi)$ is the solution of the evolution equation (1) with the initial value $d(s)u(T', \varphi)$. If $T = +\infty$ then $u_s^{T'}(\cdot)$ is defined on $[0, +\infty)$.

Theorem 1 yields several corollaries, which expand the local properties of the solutions making them global. For instance, Theorem 1 and Proposition 2 immediately imply the following

Corollary 1 (On Existence and Prolongation of Solutions): Let the operator $f : \Omega \subset \mathbf{B} \to \mathbf{B}$ be a $\mathbf{d}$-homogeneous operator on a set $\Omega \subset \mathbf{B}$. If there exists a set $M \subset \Omega$ such that $\bigcup\limits_{s \in \mathbb{R}} d(s)M = \Omega$ and the Cauchy problem (1), (2) has a solution $u(\cdot,\varphi) : [0, T_{\varphi}) \to \mathbf{B}$ for any $\varphi \in M$ then it has a solution for any $\varphi \in \Omega$. Moreover, if $T_{\varphi} = +\infty$ for all solutions with $\varphi \in M$ then all solutions of the evolution (1) with $\varphi \in \Omega$ exist on $\mathbb{R}_+$.

D. Stability of Homogeneous Evolution Equations

Homogeneity is a supporting tool for analysis of the qualitative behavior of the system. For example, it helps to classify the convergence rate. However, the homogeneity arguments cannot be used without some conventional stability analysis.

Recall that the solution $u_0 : \mathbb{R}_+ \to \Omega$ of the evolution equation (1) is said to be **Lyapunov stable** if there exists a monotone increasing function $\sigma : [0, +\infty) \to [0, +\infty)$, $\sigma(0) = 0$ and a number $h \in \mathbb{R}_+$ such that $\|u(t,\varphi) - u_0(t)\| \le \sigma(\|\varphi - u_0(0)\|)$ for all $t \in \mathbb{R}_+$ and for any $\varphi \in \Omega : \|\varphi - u(0)\| < h$.

For **asymptotic stability** of the solution u_0 we need to ask additionally the **local attractivity** of u_0, i.e. $u(t,\varphi) \to u_0(t)$ as $t \to +\infty$ if $\varphi \in \Omega : \|\varphi - u_0(0)\| < h$, where the number $h \in \mathbb{R}_+$ defines the domain of attraction.

The solution $u_0 : \mathbb{R}_+ \to \Omega$ of the evolution equation (1) is said to be **uniformly asymptotically stable** if it is asymptotically stable and, in addition, for any $r \in (0, h)$ and any

$\varepsilon \in (0, r)$ there exists $\tilde{T} \in \mathbb{R}_+$ such that $\|u(t,\varphi) - u_0(t)\| < \varepsilon$ for all $t > \tilde{T}$ and all solutions of the Cauchy problem (1), (2) with $\varphi \in \Omega : \|\varphi - u_0(0)\| < r$.

We refer the reader to [38] for more explanations of different stability properties of evolution equations in Banach spaces.

Below we study the stability property of the zero solution (i.e $u_0(\cdot) = 0$) of the equation (1). Note that the conditions $0 \in \Omega$ and $f(0) = 0$ guarantee existence of the zero solution.

Corollary 2 (On Expansion of Attraction Domain): Let $f : \Omega \subset \mathbf{B} \to \mathbf{B}$ be $\mathbf{d}$-homogeneous operator and $0 \in \Omega$, $f(0) = 0$. If the zero solution of the evolution equation (1) is locally attractive, then it is globally attractive (i.e. $h = +\infty$). If, in addition, the zero solution is Lyapunov stable then it is globally asymptotically stable.

The presented corollary can be easily extended to the case of uniform asymptotic stability. Homogeneity also simplifies a finite-time stability [39], [40] analysis of the zero solution of evolution equations. **Finite-time stability** (also known as Super-Stability [41] for infinite dimensional systems) is the version of the asymptotic stability with a finite reaching time of the stable solution, i.e. for any $\varphi \in \Omega\backslash\{0\} : \|\varphi - u_0(0)\| < h$ there exists $T \in \mathbb{R}_+$ such $\|u(t,\varphi) - u_0(t)\| = 0$ for all $t \geq T$.

To the best of our knowledge, the next property of homogeneous systems has been discovered by V. I. Zubov in 1957 for ordinary differential equations and standard (Euler) homogeneity [1, Corollary 3, Page 110].

Theorem 2 (On finite-time stability): Let $f : \Omega \subset \mathbf{B} \to \mathbf{B}$ be $\mathbf{d}$-homogeneous operator of **negative degree** $\nu < 0$ and $0 \in \Omega$, $f(0) = 0$. If the zero solution of the evolution equation (1) is uniformly asymptotically stable then it is globally **finite-time stable**.

Proof: By Corollary 2 the local uniform asymptotic stability of homogeneous evolution equation implies the global one and we have $\|u(t,\varphi)\| \leq 1$ for $t \geq \tilde{T}$, where a finite non-negative number $\tilde{T}$ exists for each $\varphi \in \Omega$.

Proposition 1 implies existence of a number $s > 0$ such that $\|\mathbf{d}(s)\|_{\mathcal{L}} = c > 1$. Since the zero solution is uniformly asymptotically stable then there exists $T' > 0$ such that $\|u(t,\varphi)\| \leq 1/c$ for all $t \geq T'$ and any $\varphi \in \Omega : \|\varphi\| \leq 1$, where $u(t,\varphi)$ is a solution of (1), (2).

Let us introduce the following notation:

- $\Delta T_0 = T'$ and $\Delta T_i = e^{\nu s}\Delta T_{i-1}$ for $i = 1, 2, ...$;
- $T_0 = 0$ and $T_i = T_{i-1} + \Delta T_{i-1}$ for $i = 1, 2, ...$;
- $x_i = u(T_i, \varphi)$ for $i = 1, 2, ...$

Obviously, since $u(T_1,\varphi) = x_1$ then $\|x_1\| \leq 1/c$. By Theorem 1 we have that $u_1(t) = \mathbf{d}(s)u(T_1 + e^{\nu s}t, \varphi)$ is also solution of (1) defined on $\mathbb{R}_+$. Moreover, $u_1(0) = \mathbf{d}(s)u(T_1,\varphi) = \mathbf{d}(s)x_1$ and $\|u_1(0)\| \leq \|\mathbf{d}(s)\|_{\mathcal{L}} \cdot \|x_1\| \leq 1$. In this case, the uniform asymptotic stability implies

$$\|u_1(T')\| = \|\mathbf{d}(s)u(T_1 + e^{\nu s}T', \varphi)\| =$$

$$\|\mathbf{d}(s)u(T_1 + \Delta T_1, \varphi)\| = \|\mathbf{d}(s)x_2\| \leq 1/c$$

and $\|\mathbf{d}(2s)x_2\| \leq \|\mathbf{d}(s)\|_{\mathcal{L}}\|\mathbf{d}(s)x_2\| \leq 1$.

Repeating the same consideration by induction we obtain $\|\mathbf{d}(is)x_i\| \leq 1$ and

$$\|u(T_i,\varphi)\| = \|\mathbf{d}(-is)\mathbf{d}(is)x_i\| \leq \|\mathbf{d}(-is)\|_{\mathcal{L}} \to 0$$

as $i \to \infty$ due to the limit property of the dilation.

Evidently, $\Delta T_i = T'e^{i\nu s}$ and for $\nu < 0$ we obtain

$$T_i = T' \sum_{n=0}^{i-1} e^{n\nu s} \to \tfrac{T'}{1-e^{\nu s}} \text{ as } i \to \infty.$$

In other words, $u(t,\varphi) \to 0$ as $t \to \tilde{T} + \frac{T'}{1-e^{\nu s}}$, where $s \in \mathbb{R}_+$ is such that $\|\mathbf{d}(s)\|_{\mathcal{L}} = c > 1$ and $\varphi \in \Omega$. ■

This theorem is very useful for qualitative stability analysis of homogeneous systems. Indeed, finite-time stability can be predicted by means of negative degree of homogeneity. Moreover, finite-time stabilizing control design can done using homogeneous feedback design with negative degree. The related problems appear in control theory (for models represented by ordinary differential equations see, for example, [11], [12], [13], [16]).

Remark 2: Note that finite-time blow-up of all solutions of homogeneous evolution equation (1) with positive degree can be proven by analogy with Theorem 2 under some additional condition on uniform divergence of solutions.

Let us present some examples of finite-time stable evolution equations, which are homogeneous with negative degree.

• **Fast Diffusion Equation**. The equation of the form

$$\frac{\partial u}{\partial t} - \Delta\left(u^{\alpha}\right) = 0, \quad \alpha \in (0,1),$$

where Δ is the Laplace operator, v is a scalar nonnegative function of time $t \in \mathbb{R}_+$ and the space variables $x \in \mathbb{R}^n$, is known as fast diffusion equation [42], [43], [44], which occurs in modeling of plasmas. The considered equation is studied with the homogeneous Dirichlet conditions $u(t,x) = 0$ for $x \in \partial M$, where $M \in \mathbb{R}^n$ is a bounded connected domain with a smooth boundary. The considered system was studied in [44] under the assumption $\frac{[n-2]_+}{n+2} < \alpha < 1$, where $[\cdot]_+$ is the projector to $\mathbb{R}_+ \cup \{0\}$, which was required for existence of a weak solution for any nonnegative initial condition $u(0,x) = u_0(x)$, $x \in M$, where $u_0 \in L^p(M,\mathbb{R})$, $p \geq 1$. Finite-time stability of fast diffusion equation has been proven in [44].

Let us show that the system is $\mathbf{d}$-homogeneous of negative degree. Indeed, it has an operator $f : D \subset L^1(M,\mathbb{R}) \to L^1(M,\mathbb{R})$ defined by $f(u) = \Delta\left(u^{\alpha}\right)$ using weak derivatives, where $D = L^1(M,\mathbb{R}_+)$. The operator f is $\mathbf{d}$-homogeneous of negative degree $\alpha - 1$ on $L^1(M,\mathbb{R})$ with the uniform dilation $\mathbf{d}(s) = e^s$, where $s \in \mathbb{R}$. Indeed, $f(d(s)u) = \Delta\left((e^s u)^{\alpha}\right) = e^{\alpha s}\Delta(u^s) = e^{(\alpha-1)s}d(s)f(u)$ for any $s \in \mathbb{R}$.

• **Finite-time Stabilization of Heat Equation on Semi-Axis**. The simplest example of distributed homogeneous control design can be presented for heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g,$$

where u is a scalar function of time $t > 0$ and space $x > 0$ variables, g is a distributed control input. The heat equation is studied with the homogeneous Dirichlet conditions $u(t,0) = 0$. The simplest finite-time stabilizing homogeneous distributed feedback can be designed as follows

$$g = -u^{1/3}.$$

In the paper [45] the finite-time stability of the considered system has been proven. Let us show that the corresponding evolution equation (1) with $f(u) = \frac{\partial^2 u}{\partial x^2} - u^{1/3}$

is $\mathbf{d}$-homogeneous for the dilation group $(\mathbf{d}(s)u)(x) = e^{3s}u(e^{-s}x)$. Indeed, $[f(d(s)u)](x) = \left[e^{3s}u(e^{-s}x)\right]'' - e^{s}u^{1/3}(e^{s}x) = e^{-2s}[d(s)f(u)](x)$. Since homogeneity degree is negative, then uniform asymptotic stability of the zero solution will imply its finite-time stability.

IV. Conclusions

The notion of $\mathbf{d}$-homogeneous evolution equation introduced in this paper can be considered as a certain analog of geometric homogeneity well-known for finite-dimensional vector fields. The obtained results about stability and scalability properties of homogeneous evolution equations provide a background required for expansion of homogeneous methods (like Input-to-State Stability) to evolution equations in a Banach space. The $\mathbf{d}$-homogeneity in this case will play an important role in the analysis and design of fast (finite-time) and robust control and observation algorithms for distributed parameter systems. In particular, Implicit Lyapunov Function method for finite-time stabilizing control design [25] can be extended to homogeneous evolution equations. We consider this as an important problem for future research.

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